

B.Sc. DEGREE (C.B.C.S.S.) EXAMINATION, OCTOBER 2016

Fifth Semester

Core Course—ABSTRACT ALGEBRA

(Common for Model I and Model II B.Sc. Mathematics)

[2013 Admission onwards]

Time : Three Hours

Maximum : 80 Marks

Part A

*Answer all questions.**Each question carries 1 mark.*

1. On Z^+ define $a * b = a | b$. Is $*$ a binary operation on Z^+ .
2. Define a group.
3. Is the group $\langle 6z, + \rangle$ cyclic.
4. Every group of order ≤ 4 is cyclic.

Write True or False :

5. What is the symmetric group on n letters ? How many elements such a group has ?
6. A subgroup of a group is a left coset of itself. Write True or False.
7. Define Kernel of a homomorphism $\phi : G \rightarrow G^1$.
8. Find all the units of Z_5 .
9. What is the characteristic of the ring Z_u ?
10. Define a prime ideal.

(10 × 1 = 10)

Part B

*Answer any eight questions.**Each question carries 2 marks.*

11. Show that for all $a, b \in G$, where G is a group, $(a * b)^1 = b^1 * a^1$, a^1 denote the inverse of a .
12. Let $*$ be defined on the set \mathbb{R}^* of non-zero real numbers by letting $a * b = a | b$. Check whether $*$ gives a group structure on \mathbb{R}^* .

Turn over

13. Show that if H and K are subgroups of an abelian group G , then $\{Gk \mid G \in H, k \in K\}$ is a subgroup of G .
14. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$. Compute $Z\sigma$.
15. Find all orders of subgroups of Z_8 .
16. Prove that every group of prime order is cyclic.
17. Compute the factor group $(z_4 \times z_6) / \langle (0,2) \rangle$.
18. Find the number of elements in the cyclic subgroup of Z_{30} , generated by 25.
19. Let ϕ be a homomorphism of a group G onto a group G^1 . Show that if G is abelian, G^1 is also abelian.
20. Prove that every field is an integral domain.
21. Show that the characteristic of an integral domain D must be either 0 or a prime P .
22. Prove that a ring homomorphism $\phi: \mathbb{R} \rightarrow \mathbb{R}^1$ is a one-to-one map if and only if $\text{Ker}(\phi) = [0]$.

(8 × 2 = 16)

Part C

*Answer any six questions.
Each question carries 4 marks.*

23. Let G be a group and $a \in G$. Show that $H = \{a^n, n \in \mathbb{Z}\}$ is a subgroup of G and is the smallest subgroup of G that contains a .
24. If G is a group with binary operation $*$, show that the left and right cancellation laws hold in G .
25. Prove that every cyclic group is abelian.
26. Show that intersection of normal subgroups of G is again a normal subgroup of G .
27. Prove that μ is a maximal normal subgroup of G if and only if G/μ is simple.
28. If $\phi: G \rightarrow G^1$ is a group homomorphism, show that $\text{Ker}(\phi)$ is a normal subgroup of G .

29. If \mathbb{R} is a ring with identity. Show that for any $a, b \in \mathbb{R}$:
- (i) $0a = a0 = 0$.
 - (ii) $a(-b) = (-a)b = -(ab)$.
 - (iii) $(-a)(-b) = ab$.
30. Let N be an ideal of a ring \mathbb{R} . Show that $r = \mathbb{R} \rightarrow \mathbb{R}/N$ given by $r(x) = x + N$ is a ring homomorphism with Kernel N .
31. Let \mathbb{R} be a commutative ring and $a \in \mathbb{R}$. Show that $I_a = \{x \in \mathbb{R} | ax = 0\}$ is an ideal of \mathbb{R} .

(6 × 4 = 24)

Part D

*Answer any two questions.
Each question carries 15 marks.*

32. (a) Prove that if $n \geq 2$, the collection of all even permutations of $\{1, 2, \dots, n\}$ forms a subgroup of order $n!/2$ of the symmetric group S_n .
- (b) Let G and G^1 be groups and let $\phi: G \rightarrow G^1$ be a one-to-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. Prove that $\phi[G]$ is a subgroup of G^1 and ϕ provides an isomorphism of G with $\phi[G]$.
33. (a) Let G be a cyclic group with generator a . Prove that if the order of G is infinite, then G is isomorphic to $\langle \mathbb{Z}, + \rangle$. Also show that if G has finite order n , then G is isomorphic to $\langle \mathbb{Z}_n, + \rangle$.
- (b) Prove that if H is a subgroup of a finite group G , then the order of H is a divisor of order of G .
34. (a) Show that every finite integral domain is a field.
- (b) State and prove fundamental homomorphism theorem for groups.

Turn over

35. (a) Let H be a subring of the ring \mathbb{R} . Show that multiplication of additive cosets of H is well defined by the equation $(a+H)(b+H) = ab+H$ if and only if $ah \in H$ and $hb \in H$ for all $a, b \in \mathbb{R}$ and $h \in H$.

(b) Show that if \mathbb{R} is a ring with unity and N is an ideal of \mathbb{R} such that $N \neq \mathbb{R}$, then \mathbb{R}/N is a ring with unity.

(2 × 15 = 30)